Resolvent Formulas, Special and General (Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics)

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Abstract

Role of resolvent formulas in determining perturbed generators is reviewed and some results presented in [6], [7], and [5] are summarized. Given a particular selfadjoint operator H_0, all selfadjoint operators are parameterized through resolvent formulas by some objects to be specified. Two approaches, one by perturbation theory and the other by extension theory, are presented and their close relation is discussed.

1. Introduction In the present note, we shall look at the so-called resolvent formulas from a general viewpoint. It is not directly connected with evolution equations, the main theme of this conference, inasmuch as no time variable t appears. Rather, it is related to methods of constructing perturbed generators (selfadjoint operators) and tries to put these constructions in one perspective through a general form of resolvent formulas. Two approaches to resolvent formulas, perturbation theoretic one and extension theoretic one, will be presented and shown that they are almost equivalent. We may emphasize that the perturbation approach in this note is in the sphere of influence of Tosio Kato's principle, in particular of [3].

This note is based on joint works with H. Nagatani ([6], [7]) and P. Kurasov ([5]) and we shall leave most of the proof to these works.

2. Resolvent formulas We shall consider two selfadjoint operators H_0 and H and put for simplicity

$$R_0(z) = (H_0 - z)^{-1}, \quad R(z) = (H - z)^{-1}, \tag{1}$$

$$\Delta(z) = R(z) - R_0(z). \tag{2}$$

When $H = H_0 + V$, two resolvents $R(z)$ and $R_0(z)$ are related as

$$\Delta(z) = -R(z)VR_0(z) = -R_0(z)VR(z), \quad z \in \rho(H_0) \cap \rho(H). \tag{3}$$

Here, $\rho(H)$ denotes the resolvent set of H. This formula is called the *second resolvent equation.* In this form the formula is symmetric with respect to H_0 and H. However, in the situation that H_0 is regarded as the unperturbed operator and V a perturbation, it is desirable not to have $R(z)$ on the right hand side so that a relation like (3) expresses $R(z)$ in terms of *known quantities* $R_0(z)$ and V.

The simplest and well-known example is the perturbation of rank one:

$$H = H_0 + c(\cdot, \varphi)\varphi, \quad c \in \mathbf{R}, \quad \|\varphi\| = 1. \tag{4}$$

In this case a direct computation shows that

$$\Delta(z) = -\frac{1}{D(z)}(\,\cdot\,, R_0(\bar{z})\varphi)\, R_0(z)\varphi, \tag{5}$$

where $D(z)$ is a function defined as

$$D(z) = 1 + c(R_0(z)\varphi, \varphi), \quad z \in \rho(H_0). \tag{6}$$

As expected, (5) describes $\Delta(z)$ only by known quantities. We call generally such a formula a *resolvent formula.* (A prototype of this computation is in [2].)

Only obscure point on the right hand side of (5) is that when $1/D(z)$ makes sense, i.e. that when $D(z) = 0$. In this respect, it is known that

$$\{z \in \rho(H_0) \mid D(z) = 0\} = \sigma_p(H) \cap \rho(H_0), \tag{7}$$

where $\sigma_p(H)$ denotes the point spectrum (the set of all eigenvalues) of H. Moreover, the eigenspace is the one-dimensional space spanned by $R_0(z)\varphi$.

Another example in which $\Delta(z)$ is of rank one occurs in the theory of extension of a symmetric operator with the deficiency indices $(1,1)$. Let A be a closed symmetric operator with the deficiency indices $(1,1)$ and let $\mathcal{M} = (A - i)\mathcal{D}(A)^{\perp}$ be the deficiency subspace of A. Here, $\mathcal{D}(A)$ denotes the domain of A. Suppose that $\mathcal{M}$ is spanned by φ, $\|\varphi\| = 1$. Let H_0 be one particular selfadjoint extension of A. Then, other selfadjoint extensions of A is parameterized by a real parameter γ so that the following resolvent formula holds:

$$\Delta(z) = -\frac{1}{\gamma + q(z)}(\,\cdot\,, (H_0 + i)R_0(\bar{z})\varphi)\,(H_0 - i)R_0(z)\varphi, \tag{8}$$

where $q(z)$ is the function defined as

$$q(z) = ((1 + zH_0)R_0(z)\varphi, \varphi). \tag{9}$$

Again relation (7) holds with $D(z)$ replaced by $\gamma + q(z)$. (8) is celebrated *Krein's formula* and there are many subsequent generalizations ([8], [1], and others cited in [5]).

Let us proceed to a more general situation. From (3) it follows immediately that

$$\Delta(z) = -R_0(z)(1 + VR_0(z))^{-1}VR_0(z). \tag{10}$$

This is valid if V is bounded, or more generally, relatively bounded with respect to H_0 (i.e. $\mathcal{D}(V) \supset \mathcal{D}(H_0)$).

In all of these examples the perturbed operator H has been defined before resolvent formulas are discussed at all. In more delicate problems, however, the perturbed operator itself is defined by the resolvent formula. A notable example occurred in the theory of smooth perturbations due to Kato ([3]). There, the perturbation is given formally as a factorized form $V = B^*A$, but only operators $AR_0(z)$ and $BR_0(z)$ are rigorously defined and one cannot define the perturbed operator $H(\kappa) = H_0 + \kappa B^*A$ directly. Under hypotheses which express the smoothness of A and B with respect to H_0 Kato defined $R(z, \kappa)$ by

$$R(z,\kappa) = R(z) - \kappa[R(z)B^*](1 + \kappa Q(z))^{-1}AR(z), \tag{11}$$

where $Q(z)$ is an operator valued function defined as

$$Q(z) = [AR(z)B^*]^{\text{closure}}, \tag{12}$$

and proceeded to prove that $R(z, \kappa)$ is the resolvent of a closed operator $H(\kappa)$ for complex κ with sufficiently small $|\kappa|$. Thus, the resolvent formula is a key to *define* perturbed generators.

It is seen in these examples that, given H_0 and a class of perturbations, perturbed generators of the corresponding class are *parameterized* by a suitable object, in (5) and (8) by complex numbers c and γ, respectively, and in (11) by pairs $\{A, B\}$ of operators. Another common point is the appearance of characteristic (operator valued) functions which determine singular points of $R(z)$. They are $D(z)$ in (6), $\gamma + q(z)$ in (8), and $1 + \kappa Q(z)$ in (11).

In what follows we shall pursue this approach to an extreme and parameterize all selfadjoint operators by suitable objects. In the perturbation approach they are bounded operators from $\mathcal{D}(H_0)$ to $\mathcal{D}(H_0)^*$ and in the extension theoretic approach they are pairs $\{\mathcal{M}, \gamma\}$ of a subspace $\mathcal{M}$ and a selfadjoint operator in $\mathcal{M}$.

3. $\mathcal{H}_{-2}$–perturbation We shall work in a Hilbert space $\mathcal{H}_0$ and fix a selfadjoint operator H_0. Let $\mathcal{H}_2 = \mathcal{D}(H_0)$ with the graph norm and let $\mathcal{H}_{-2} = \mathcal{H}_2^*$, the adjoint space of $\mathcal{H}_2$. As usual we denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the set of all bounded operators from $\mathcal{X}$ to $\mathcal{Y}$. We regard the unperturbed resolvent either as $R_0(z) \in \mathcal{L}(\mathcal{H}_{-2}, \mathcal{H}_0)$ or as $R_0(z) \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_2)$. The space $\mathcal{H}_2$ is used in [4] to discuss a generalized rank one perturbation formally given as

$$H = H_0 + c(*, \varphi)\varphi, \quad \varphi \in \mathcal{H}_{-2}. \tag{13}$$

The following theorem is an outcome of our effort to generalize it and uses a subset of $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_{-2})$ to parameterize all selfadjoint operators.

For $T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_{-2})$ we consider the following two conditions:

$$\mathcal{N}(1 - TR_0(i)) = 0, \quad \text{where} \quad 1 - TR_0(i) \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_{-2}), \tag{14}$$

$$T - T^* = -2iT(H_0^2 + 1)^{-1}T^* = -2iT^*(H_0^2 + 1)^{-1}T, \tag{15}$$

where $\mathcal{N}(T)$ denotes the nullspace (kernel) of T. We call (14) *admissibility condition* and (15) *selfadjoint condition.*

Theorem 1 *Let H_0 be fixed. Then the relation*

$$\Delta(i) \equiv R(i) - R_0(i) = -R_0(i)TR_0(i) \tag{16}$$

determines a bijective correspondence between the set of all $T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_{-2})$ satisfying conditions (14), (15) *and the set of all selfadjoint operators H in $\mathcal{H}$. Furthermore, H and H_0 satisfy the following resolvent formula:*

$$\Delta(z) = -R_0(z)(1 + (z - i)TR_0(z)R_0(i))^{-1}TR_0(z). \tag{17}$$

The approach expressed in Theorem 1 is referred to as *$\mathcal{H}_{-2}$-perturbation.* A proof of Theorem 1 is found in [7].

Hereafter we denote by $H(T)$ the operator H determined by (16).

Let us compare (17) with (10). If one replaces V by T and $R_0(z)$ by $(z-i)R_0(z)R_0(i)$ in (10), then one arrives at (17). A crucial point is that, as $T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_{-2})$, one need to have the iterated resolvent $W(z,i) = (z - i)R_0(z)R_0(i) \in \mathcal{L}(\mathcal{H}_{-2}, \mathcal{H}_2)$ to couple with $T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_{-2})$ to obtain an operator acting in one space $\mathcal{H}_{-2}$. Note that in $\mathcal{L}(\mathcal{H}_{-2}, \mathcal{H}_2)$ $W(z,i)$ cannot be expressed as $R_0(z) - R_0(i)$.

When $\mathcal{D}(H) = \mathcal{D}(H_0)$, then $V = H - H_0 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_0)$ and $H = H_0 + V$. This case is called the *additive perturbation.* It is seen that T and V is related as

$$T = (1 + VR_0(i))^{-1}V, \quad V = (1 - TR_0(i))^{-1}T. \tag{18}$$

Using the first relation of (18) in (17) we recapture (10).

Finally, we observe a useful relation between Theorem 1 and the extension theory of symmetric operators. Let A be a closed symmetric operators in $\mathcal{H}_0$ and let H_0 and H are selfadjoint extensions of A. A is said to be the *maximal common restriction* of H_0 and H if

$$\{u \in \mathcal{D}(H) \cap \mathcal{D}(H_0) \,|\, Hu = H_0u\} = \mathcal{D}(A). \tag{19}$$

Theorem 2 *Let A be a closed symmetric operator in $\mathcal{H}_0$. Fix an extension H_0 of A and construct $H(T)$ by Theorem 1. Then $H(T)$ is a selfadjoint extention of A if and only if $\mathcal{N}(T) \supset \mathcal{D}(A)$. A is a maximal common restriction of H_0 and $H(T)$ if and only if $\mathcal{N}(T) = \mathcal{D}(A)$.*

For the purpose of illustration let us take up a trivial example of first order differential operators which is formally expressed as

$$H = -i\frac{d}{dx} + \alpha\delta(x). \tag{20}$$

We work in $L^2(\mathbf{R})$. Let

$$H_0 = -i\frac{d}{dx}, \quad \mathcal{D}(H_0) = H^1(\mathbf{R}), \tag{21}$$

where $H^1(\mathbf{R}) \equiv H^1$ is the Sobolev space. Let $H^{-1} = H^{-1}(\mathbf{R}) = (H^1)^*$.

Let $\mathbf{R}_+ = (0, \infty)$ and $\mathbf{R}_- = (-\infty, 0)$. Every $f \in H^1(\mathbf{R}_\pm)$ has the boundary value $f(0\pm)$. Selfadjoint operators corresponding to formal expression (20) is taken to be selfadjoint extensions of $H_0|_{C_0^\infty(\mathbf{R}\setminus\{0\})}$. It is well-known that such extensions are given by H_t, $0 \le t < 2\pi$, whose domain is given as

$$\mathcal{D}(H_t) = \{\, f \in H^1(\mathbf{R}_-) \oplus H^1(\mathbf{R}_+) \mid f(0+) = e^{it} f(0-)\}. \tag{22}$$

In our framework this result may be derived as follows. By Theorem 2 such extensions are given by $H(T)$ with T satisfying (i) $T \in \mathcal{L}(H^1, H^{-1})$; (ii) relations (14), (15); (iii) $\mathcal{N}(T) \supset C_0^\infty(\mathbf{R} \setminus \{0\})$. From (i) and (iii) it follows that T is a rank one operator of the form $T = a(\,\cdot\,, \delta)\delta$ (note that $\delta \in H^{-1}$). Relation (14) is automatically satisfied because the range of T does not contain non-zero elements of L^2. By an easy computation it is shown that (15) is satisfied if and only if $|a + i| = 1$. Such a is expressed as $a(t) = e^{i(t+\pi/2)} - i$, $0 \le t < 2\pi$. Thus, we have seen that all selfadjoint extensions of $H_0|_{C_0^\infty(\mathbf{R}\setminus\{0\})}$ is given by $H_t = H(T(t))$ with $T(t) = a(t)(\,\cdot\,, \delta)\delta$, $0 \le t < 2\pi$. From (16) it follows that $\mathcal{D}(H_t) = (1 - R_0(i))H^1$ and hence $\mathcal{D}(H_t) = \{\, g - a(t)g(0)R_0(i)\delta \mid g \in H^1(\mathbf{R})\,\}$. (22) follows from this at once, because $(R_0(i)\delta)(x) = ie^{-x}\Theta(x)$, $\Theta(x)$ being the Heaviside function.

4. Relation to extension theory After the completion of the preprint version of [7] Pavel Kurasov (private conversation) pointed out that resolvent formula (17) must have close connections with Krein's formula. This issue has been investigated in [5].

In the extension theory one usually fixes a particular selfadjoint extensions of a closed symmetric operator A and tries to characterize all selfadjoint extensions of A. (An example of rank one case was discussed in paragraph 1 (see (8)).) Looking from a different viewpoint we may proceed as follows. Fix a selfadjoint operator H_0. Let $\mathcal{N}$ be a closed subspace of $\mathcal{D}(H_0)$ with the graph norm and consider all selfadjoint extensions H of $A = H_0|_{\mathcal{N}}$ such that A is the maximal common restriction of H_0 and H. If we vary $\mathcal{N}$ and collect all such extensions H, we would obtain all selfadjoint operators. Instead of $\mathcal{N}$ we use the deficiency subspace $\mathcal{M} = (A - i)\mathcal{N}^\perp$. Then, our result is summarized as follows.

A pair $\{\mathcal{M}, \gamma\}$ of a closed subspace $\mathcal{M}$ of $\mathcal{H}_0$ and a selfadjoint operator γ in $\mathcal{M}$ is called *admissible* if

$$\mathcal{N}\left(\frac{1}{H_0+i} - \frac{1}{\gamma+i}P_{\mathcal{M}}\right) = \{0\}. \tag{23}$$

Theorem 3 *Let H_0 be fixed. Then the relation*

$$\Delta(i) \equiv R(z) - R_0(z) = -\frac{H_0+i}{H_0-i}\frac{1}{\gamma+i}P_{\mathcal{M}}, \tag{24}$$

where $P_{\mathcal{M}}$ is the orthogonal projection onto M, determines a bijective correspondence between the set of all admissible pairs $\{\mathcal{M}, \gamma\}$ and the set of all selfadjoint operators H in $\mathcal{H}_0$. Furthermore, H and H_0 satisfy the following resolvent formula:

$$\Delta(z) = -(1+(z+i)R_0(z))(\gamma+Q(z))^{-1}P_{\mathcal{M}}(1+(z-i)R_0(z)), \tag{25}$$

$$Q(z) = P_{\mathcal{M}}(1+zH_0)R_0(z)|_{\mathcal{M}}. \tag{26}$$

Hereafter, we denote by $H(\mathcal{M}, \gamma)$ the operator H determined by (24). Relations of Theorem 4 to the extension theory may be more clearly seen in the following theorem.

Theorem 4 *Let $\mathcal{N}$ be a closed subspace of $\mathcal{D}(H_0)$ with the graph norm and put $\mathcal{M}_{\mathcal{N}} = \{(H_0-i)\mathcal{N}\}^{\perp}$. Then, $H(\mathcal{M}, \gamma)$ is a selfadjoint extension of $H_0|_{\mathcal{N}}$ if and only if $\mathcal{M} \subset \mathcal{M}_{\mathcal{N}}$. $H_0|_{\mathcal{N}}$ is the maximal common restriction of H_0 and $H(\mathcal{M}, \gamma)$ if and only if $\mathcal{M} = \mathcal{M}_{\mathcal{N}}$.*

Finally, we shall discuss relations between Theorems 2 and 3, or relations between T and $\{\mathcal{M}, \gamma\}$. Suppose $H(T) = H(\mathcal{M}, \gamma)$. Then, it is immediately seen from (16) and (24) that

$$R_0(-i)TR_0(i) = (\gamma+i)^{-1}P_{\mathcal{M}}. \tag{27}$$

Hence, T is expressed by $\{\mathcal{M}, \gamma\}$ as

$$T = (H_0+i)(\gamma+i)^{-1}P_{\mathcal{M}}(H_0-i). \tag{28}$$

On the other hand it follows from (27) that $\mathcal{M} = \{\mathcal{N}(TR_0(i))\}^{\perp}$ and we have

$$(\gamma+i)^{-1} = R_0(-i)TR_0(i)P|_{\mathcal{M}}. \tag{29}$$

It can be shown that T and $\{\mathcal{M}, \gamma\}$ related in this way satisfy respective admissibility condition at the same time. Thus, we could derive Theorem 1 and Theorem 3 from each other. We think, however, that, since ways of approach in the perturbation theory and the extension theory are rather different, it may be worthwhile to develop them separately.

Finally, we remark that Theorem 1 can be extended to the case that $H(T)$ is not selfadjoint (see [6]).

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